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On schizosymmetric superfields and $sl(2/1,\mathbb{C})_{\mathbb{R}}$ supersymmetry

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Superfield expansions over four-dimensional graded spacetime (x^μ, θ^ν), with Minkowski coordinates x extended by vector Grassmann variables θ, are investigated. By appropriate identification of the physical Lorentz algebra in the even and odd parts of the superfield, a typology of 'schizofields' containing both integer and half-integer spin fields is established. For two of these types, identified as 'gauge potential'-like and 'field strength'-like schizofields, an $sl(2/1,\mathbb{C})_{\mathbb{R}}$ supersymmetry at the component field level is demonstrated. Prospects for a schizofield calculus, and application of these types of fields to the particle spectrum, are adumbrated.

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1 Introduction

In a previous paper[1] a new paradigm of 'schizosymmetry' for superfield expansions was proposed. The principle was based on the assignment of symmetry algebras (of either spacetime or internal supersymmetry) of the type

$$T^{phys} = T^{odd}\mathbb{P}^{o} + T^{even}\mathbb{P}^{e} \tag{1}$$

whereby physical generators may be assigned differently on even and odd parts of the superfield. In [1] the implications of (1) were explored for spacetime symmetries in four dimensions, and candidate superfields carrying both integer and half-integer spin component fields were identified.

In the present work the analysis of [1] is extended by giving the most general schizofield expansions over graded spacetime in four dimensions, with coordinates x^{μ} extended by 'vector' Grassmann variables θ^{ν}, with physical spin assignments which are compatible with (1) and with spin-statistics (§2 below). Further, it is shown in §3 that each of the three classes identified, types I, II, III together with their Grassmann parity-inverted versions $\widetilde{I}$, $\widetilde{II}$, $\widetilde{III}$ can be viewed as irreducible 16-dimensional representations of one of the super-Lorentz algebras $sl(2/1,\mathbb{C})_{\mathbb{R}}$ or $osp(3,1/4,\mathbb{R})$. For the former, 'gauge potential'-like and 'field strength'-like schizofields are identified, and the generators of $sl(2/1,\mathbb{C})_{\mathbb{R}}$ given explicitly. The prospects for a schizofield calculus, and application of these types of fields to the particle spectrum, are adumbrated in §4 which also includes concluding remarks and outlook.

Apart from antecedents in general works on supersymmetry and superfields (see references in [1]), work related to the present approach to superfields is that of [2], which also develops superfields over vector Grassmann coordinates (especially in $d \geq 5$ dimensions) extended to local symmetries. In connection with generalised 'graded Lorentz' supersymmetries, the paper [3] should be noted. Finally, although having the use of vector Grassmann coordinates in common, studies of supersymmetric quantum mechanics and the index theorem (see for example [4]) and related spinning particle models appear to be different from the present schizosymmetric superfields. The present work is partly based on [5]

2 Typology of schizofields

In order to introduce the notion of schizosymmetry in the space-time context in four dimensions, it is necessary to establish some notation on the structure of the Lorentz algebra and its representations. For completeness this is given below. This material itself is standard (see for example [6]), but it is applied here specifically to the analysis of the spin content of the types of superfields under study.

Preliminaries

Graded spacetime is taken to comprise four dimensional Minkowski space with standard coordinates $(x^{\mu}) = (x^0, x^1, x^2, x^3)$ and Lorentz metric $(\eta_{\mu\nu}) = \mathrm{diag}(+1,-1,-1,-1)$,

extended by 'vector' Grassmann coordinates $(\theta^\mu) = (\theta^0, \theta^1, \theta^2, \theta^3)$. General superfield expansions are then of the form

$$\Phi(x,\theta) = A(x) + \theta^\mu V_\mu(x) + \tfrac{1}{2}\theta^\mu\theta^\nu F_{\mu\nu}(x) + \tfrac{1}{6}\theta^\mu\theta^\nu\theta^\rho A_{\mu\nu\rho}(x) + \tfrac{1}{24}\theta^\mu\theta^\nu\theta^\rho\theta^\sigma B_{\mu\nu\rho\sigma}(x) \quad (2)$$

with the generators of Lorentz transformations represented differentially by†

$$L_{\mu\nu} = (x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}) + (\theta_\mu \frac{\partial}{\partial \theta^\nu} - \theta_\nu \frac{\partial}{\partial \theta^\mu}).$$

That such superfields may also carry spinors is evident from the fact that representations of the Dirac algebra are available,

$$\begin{aligned} \gamma^\pm_\mu &= (\theta_\mu \pm \frac{\partial}{\partial\theta^\mu}), \\ \{\gamma^\pm_\mu, \gamma^\pm_\nu\}_+ &= \pm 2\eta_{\mu\nu}, \\ \{\gamma^+_\mu, \gamma^-_\nu\}_+ &= 0. \end{aligned} \quad (3)$$

with, for example $\gamma^\pm_5 = i\gamma^\pm_0\gamma^\pm_1\gamma^\pm_2\gamma^\pm_3$ given by

$$\gamma^\pm_5 = \tfrac{i}{24}\epsilon^{\mu\nu\rho\sigma}(\theta_\mu\theta_\nu\theta_\rho\theta_\sigma \pm 4\theta_\mu\theta_\nu\theta_\rho\partial_\sigma + 6\theta_\mu\theta_\nu\partial_\rho\partial_\sigma \pm 4\theta_\mu\partial_\nu\partial_\rho\partial_\sigma + \partial_\mu\partial_\nu\partial_\rho\partial_\sigma). \quad (4)$$

The two (anti-commuting, *graded*) representations of the Dirac algebra lead to two inequivalent assignments of the spin part of the Lorentz algebra

$$\begin{aligned} L^\pm_{\mu\nu} &= \pm\tfrac{1}{4}[\gamma^\pm_\mu, \gamma^\pm_\nu]_-, \\ L^\pm_{\mu\nu} &= \tfrac{1}{2}(\theta_\mu \frac{\partial}{\partial\theta^\nu} - \theta_\nu \frac{\partial}{\partial\theta^\mu}) \pm \tfrac{1}{2}(\theta_\mu\theta_\nu + \frac{\partial}{\partial\theta^\mu}\frac{\partial}{\partial\theta^\nu}) \\ [L_{\mu\nu}, L_{\rho\sigma}]_- &= \eta_{\rho\nu}L_{\mu\sigma} - \eta_{\rho\mu}L_{\nu\sigma} - \eta_{\sigma\nu}L_{\mu\rho} + \eta_{\sigma\mu}L_{\nu\rho} \end{aligned}$$

In the usual way, one can effect the decomposition of the $L^\pm_{\mu\nu}$ into self-dual and anti-self dual combinations,

$$L^{L/R}_{\mu\nu} = \frac{1}{2}(L_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}L_{\rho\sigma}) \equiv \frac{1}{2}(L_{\mu\nu} \pm i\tilde{L}_{\rho\sigma})$$

which satisfy the same commutation relations as $L_{\mu\nu}$ but mutually commute. In order to see the implications of these definitions it is convenient also to introduce the Weyl representation of the Dirac algebra. Dirac spinors are represented as $\psi = {}^T\!\big(u_a,\ \bar{v}^{\dot a}\big)$ with

$$\gamma_\mu = \begin{pmatrix} 0 & \bar\sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix}, \qquad \tfrac{1}{4}[\gamma_\mu, \gamma_\nu] = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \bar\sigma_{\mu\nu} \end{pmatrix},$$

$$\sigma_{\mu\nu a}{}^b \equiv \tfrac{1}{4}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)_a{}^b \qquad \bar\sigma^{\dot a}_{\mu\nu\dot b} \equiv \tfrac{1}{4}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu)^{\dot a}{}_{\dot b}$$

Then the self-dual parts are projected simply via

$$\begin{aligned} J_{ab} \equiv\ & \tfrac{1}{4}(\sigma^{\mu\nu}L_{\mu\nu})_{ab} \equiv \tfrac{1}{4}(\sigma^{\mu\nu}L^L_{\mu\nu})_{ab} \\ \bar J_{\dot a\dot b} \equiv\ & \tfrac{1}{4}(\bar\sigma^{\mu\nu}L_{\mu\nu})_{\dot a\dot b} \equiv \tfrac{1}{4}(\bar\sigma^{\mu\nu}L^R_{\mu\nu})_{\dot a\dot b} \end{aligned}$$

† The usual graded Leibnitz rule $\frac{\partial}{\partial\theta^\mu}\theta^\nu = \delta_\mu{}^\nu - \theta^\nu\frac{\partial}{\partial\theta^\mu}$ applies.

Table 1: Schizofield spherical harmonics: θ-polynomials covariant with respect to $SO(3,1)^+ + SO(3,1)^- \simeq sl(2)_L^+ + sl(2)_R^+ + sl(2)_L^- + sl(2)_R^-$

θ-polynomial	$\gamma_5^+\ \gamma_5^-$	$so(3,1)^+ \oplus so(3,1)^-$
$\frac{1}{2}(1+i\theta^4)$	$+\,+$	$(0,\frac{1}{2})\otimes(0,\frac{1}{2})$
$\frac{1}{2}(1-\theta^4)$	$-\,-$	$(\frac{1}{2},0)\otimes(\frac{1}{2},0)$
$\frac{1}{2}(\theta^\mu+i\tilde{\theta}^\mu)$	$-\,+$	$(\frac{1}{2},0)\otimes(0,\frac{1}{2})$
$\frac{1}{2}(\theta^\mu-i\tilde{\theta}^\mu)$	$+\,-$	$(0,\frac{1}{2})\otimes(\frac{1}{2},0)$
$\frac{1}{2}(\theta^{\mu\nu}+i\tilde{\theta}^{\mu\nu})$	$-\,-$	$(\frac{1}{2},0)\otimes(\frac{1}{2},0)$
$\frac{1}{2}(\theta^{\mu\nu}-i\tilde{\theta}^{\mu\nu})$	$+\,+$	$(0,\frac{1}{2})\otimes(0,\frac{1}{2})$

because $\tilde{\sigma}_{\mu\nu} = -i\sigma_{\mu\nu}$ and $\tilde{\bar{\sigma}}_{\mu\nu} = i\bar{\sigma}_{\mu\nu}$. The identification of $so(3,1)$ with $sl(2,\mathbb{C})_\mathbb{R}$ is completed by examining the structure of $sp(2) \sim sl(2)$ generated by $\{K_{ab} = K_{ba}; a, b = 1,2\}$,

$$[K_{ab}, K_{cd}]_- = \varepsilon_{cb}K_{ad} + \varepsilon_{ca}K_{bd} + \varepsilon_{da}K_{bc} + \varepsilon_{db}K_{ac},$$

where $\varepsilon_{12} = 1 = -\varepsilon_{21}$ and $\varepsilon_{11} = 0 = \varepsilon_{22}$. Then $sp(2,\mathbb{C})_\mathbb{R}$ is spanned by $\{K_{ab},\ K'_{ab} \equiv iK_{ab}\}$ as a real Lie algebra. In its complexification, we can define

$$J_{ab} := \tfrac{1}{2}(K_{ab} + iK'_{ab}), \quad J_{\dot{a}\dot{b}} := \tfrac{1}{2}(K_{\dot{a}\dot{b}} - iK'_{\dot{a}\dot{b}})$$

which mutually commute and satisfy the same algebra as $\{K_{ab}\}$. In practice, as is well known, it is convenient to label irreducible representations[‡] of the Lorentz algebra via the complexification $SO(3,1)_\mathbb{C} \simeq sp(2,\mathbb{C}) \simeq sl(2)_L \oplus sl(2)_R$.

Returning to the superfield expansion of $\Phi(x,\theta)$, note in the Dirac algebra

$$\gamma_5 \cdot \tfrac{1}{4}[\gamma_\mu, \gamma_\nu] = \tfrac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\tfrac{1}{4}[\gamma_\rho, \gamma_\sigma]$$

and also

$$\tfrac{1}{2}(1+\gamma_5)\cdot\begin{pmatrix}\sigma_{\mu\nu} & 0\\ 0 & \bar{\sigma}_{\mu\nu}\end{pmatrix} = \begin{pmatrix}\sigma_{\mu\nu} & 0\\ 0 & 0\end{pmatrix},$$
$$\tfrac{1}{2}(1-\gamma_5)\cdot\begin{pmatrix}\sigma_{\mu\nu} & 0\\ 0 & \bar{\sigma}_{\mu\nu}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & \bar{\sigma}_{\mu\nu}\end{pmatrix}$$

so that the $\frac{1}{2}(1\pm\gamma_5^\pm)$ projections can be used to decompose superfield components in $\Phi(x,\theta)$ with respect to $SO(3,1)^+ + SO(3,1)^- \simeq sl(2)_L^+ + sl(2)_R^+ + sl(2)_L^- + sl(2)_R^-$ and determine the complete spectrum of $(j_1,j_2)^\pm$. The results are shown in table 1[§].

Schizosymmetry - a new paradigm for superfield expansions

The principle of schizosymmetry is now implemented by the choice of physical Lorentz algebra in extended spacetime,

$$L^{phys} = L^{odd}\mathbb{P}^o + L^{even}\mathbb{P}^e \tag{5}$$

[‡]For example, for a Dirac spinor $\sim (\frac{1}{2},0)+(0,\frac{1}{2})$, 4-vector $\sim (\frac{1}{2},\frac{1}{2})$, antisymmetric tensor $\sim (1,0)+(0,1)$ and so on.

[§] Where $\theta^{\mu\nu} = \frac{1}{2}\theta^\mu\theta^\nu$, $\tilde{\theta}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\theta_\rho\theta_\sigma$, $\tilde{\theta}^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\theta_\nu\theta_\rho\theta_\sigma$, $\theta^4 = \frac{1}{24}\varepsilon^{\mu\nu\rho\sigma}\theta_\mu\theta_\nu\theta_\rho\theta_\sigma$

Table 2: Component field content of 'even' type schizofields of classes *I*, *II*, *III* (for the 'odd' types, the component field contents of the even and odd parts of the superfields are reversed).

Class	even	odd
I	$(1,0)+(0,1)+2(0,0)$	$2[(\frac{1}{2},0)+(0,\frac{1}{2})]$
II	$(\frac{1}{2},\frac{1}{2})+4(0,0)$	$2[(\frac{1}{2},0)+(0,\frac{1}{2})]$
III	$2(\frac{1}{2},\frac{1}{2})$	$2[(\frac{1}{2},0)+(0,\frac{1}{2})]$

with projection operators defined through the number operator $\mathcal{N} \equiv \theta^\mu \partial_\mu$, for example

$$\begin{aligned}
\mathbb{P}^e &= \tfrac{1}{3}(\mathcal{N}-1)(\mathcal{N}-3)(\mathcal{N}^2-4\mathcal{N}+1) \\
&= \tfrac{1}{3}(\theta^\mu\theta^\nu\theta^\rho\theta^\sigma\partial_\sigma\partial_\rho\partial_\nu\partial_\mu - 2\theta^\mu\theta^\nu\theta^\rho\partial_\rho\partial_\nu\partial_\mu + 3\theta^\mu\theta^\nu\partial_\nu\partial_\mu - 3\theta^\mu\partial_\mu + 3); \\
\mathbb{P}^o &= -\tfrac{1}{3}\mathcal{N}(\mathcal{N}-2)^2(\mathcal{N}-4) \\
&= -\tfrac{1}{3}(\theta^\mu\theta^\nu\theta^\rho\theta^\sigma\partial_\sigma\partial_\rho\partial_\nu\partial_\mu - 2\theta^\mu\theta^\nu\theta^\rho\partial_\rho\partial_\nu\partial_\mu + 3\theta^\mu\theta^\nu\partial_\nu\partial_\mu - 3\theta^\mu\partial_\mu); \\
\mathbb{P}^{(0)} &= \tfrac{1}{24}(\mathcal{N}-1)(\mathcal{N}-2)(\mathcal{N}-3)(\mathcal{N}-4) \\
&\equiv \partial_\sigma\partial_\rho\partial_\nu\partial_\mu\theta^\mu\theta^\nu\theta^\rho\theta^\sigma; \qquad (6)
\end{aligned}$$

the projection $\mathbb{P}^{(0)}$ on to the θ-independent term of the superfield will be required in §3 below. In fact, all acceptable embeddings of $L^{phys}_{\mu\nu}$ correspond to consistent reductions of the eight-dimensional even and odd projections of the schizofield into tensor and spinor representations respectively. This counting problem turns out only to have essentially three solutions, shown in table 2 as classes *I*, *II*, *III*. It is convenient also to introduce the corresponding Grassmann odd schizofield classes $\widetilde{I}$, $\widetilde{II}$, $\widetilde{III}$ corresponding to the same representation content, but with grading reversed. Note that only $so(3,1)$ irreps (j,j) or $(j_1,j_2)+(j_2,j_1)$ appear, and not cases such as $3(\frac{1}{2},0)+(0,\frac{1}{2})$ or $(\frac{1}{2},0)+2(0,0)$. Each of classes *I*, *II*, *III* corresponds to a specific assignment of Lorentz algebra of the type (5), for example

$$L^I_{\mu\nu} = L^{diag}_{\mu\nu}\mathbb{P}^e + L^+_{\mu\nu}\mathbb{P}^o \qquad (7)$$

with similar representative identifications for the other classes. Some of these exploit the semisimple nature of the Lorentz algebra in four dimensions, in that the schizosymmetric identifications of type (5) are different for $sl(2)^L$ and $sl(2)^R$. The explicit physical field content is manifested by decoupling the $^+$ and $^-$ covariance via van der Waerden notation (compare (1)), yielding

$$\begin{aligned}
\Phi_I &= A + \theta^4 B + \theta^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}(\theta^\mu + i\tilde{\theta}^\mu)\sigma_\mu^{a\dot{\alpha}}u_{a\dot{\alpha}} + \\
&\qquad \tfrac{1}{2}(\theta^\mu - i\tilde{\theta}^\mu)\bar{\sigma}^\mu_{\alpha\dot{a}}\bar{v}^{\dot{a}\alpha}, \\
\Phi_{II} &= \tfrac{1}{4}(1-i\theta^4)\varepsilon^{a\alpha}\sigma^\mu_{a\dot{\alpha}}V_\mu + \tfrac{1}{4}(1+i\theta^4)\varepsilon^{\dot{a}\dot{\alpha}}S_{\dot{a}\dot{\alpha}} \\
&\quad + \tfrac{1}{2}(\theta^{\mu\nu} - i\tilde{\theta}^{\mu\nu})\sigma^{a\alpha}_{\mu\nu}\sigma^\lambda_{a\dot{\alpha}}V_\lambda + \tfrac{1}{2}(\theta^{\mu\nu} + i\tilde{\theta}^{\mu\nu})\bar{\sigma}^{\dot{a}\dot{\alpha}}_{\mu\nu}S_{\dot{a}\dot{\alpha}} \\
&\quad + \tfrac{1}{2}(\theta^\mu + i\tilde{\theta}^\mu)\sigma^{a\dot{\alpha}}_\mu u_{a\dot{\alpha}} + \tfrac{1}{2}(\theta^\mu - i\tilde{\theta}^\mu)\bar{\sigma}^\mu_{\alpha\dot{a}}\bar{v}^{\dot{a}\alpha}, \\
\widetilde{\Phi}_{III} &= \theta^\mu V_\mu + \tilde{\theta}^\mu A_\mu \\
&\quad + \tfrac{1}{2}(\theta^{\mu\nu} + i\tilde{\theta}^{\mu\nu})\sigma^{a\alpha}_{\mu\nu}u_{a\alpha} + \tfrac{1}{2}(\theta_{\mu\nu} - i\tilde{\theta}_{\mu\nu})\bar{\sigma}^{\mu\nu}_{\dot{a}\dot{\alpha}}\bar{v}^{\dot{a}\dot{\alpha}}. \qquad (8)
\end{aligned}$$

3 Super-Lorentz symmetry $sl(2/1,\mathbb{C})_{\mathbb{R}}$

Supersymmetry?

Thus far the admissible schizosymmetric superfields have been derived from considerations of Lorentz covariance and consistency with spin-statistics, with their even and odd parts considered separately. It is natural however to look for superalgebras for which $\Phi(x,\theta)$ is an irreducible representation, and hence invoke a supersymmetric unification at least at the component field level. One such candidate is by default the superalgebra $gl(8/8)$ generated by all operators $p(\theta)q(\partial)$ for some polynomials p,q in θ^μ and derivatives, which certainly has a 16-dimensional defining representation. However, $gl(8/8)$ is likely to be too big to be a useful kinematical superalgebra in further constructions such as a schizofield calculus. More reasonably one can ask for the *smallest* superalgebra containing the Lorentz algebra for which for which $\Phi(x,\theta)$ is an irreducible representation.

Schizofields as representations of $sl(2/1,\mathbb{C})_{\mathbb{R}}$

Consider in this context $sl(2/1,\mathbb{C})_{\mathbb{R}}$, a natural supersymmetric grading of the Lorentz algebra. From the well known $sl(2/1)\supset sl(2)+U(1)$ representations (written j_z, with charge quantum number z as subscript to spin content):

$$\begin{array}{lll} \mathbf{3} & \downarrow & \frac{1}{2}_{\frac{1}{2}} + 0_1 \\ \mathbf{4} & \downarrow & \frac{1}{2}_z + 0_{z+\frac{1}{2}} + 0_{z-\frac{1}{2}} \\ \mathbf{8} & \downarrow & 1_0 + 0_0 + \frac{1}{2}_1 + \frac{1}{2}_{-1} \end{array}$$

one infers[¶] the following representations of $sl(2/1,\mathbb{C})_{\mathbb{R}}$ (choosing $z=0$):

$$\begin{array}{llll} I & (\mathbf{8},0)+(0,\mathbf{8}) & \downarrow & 2(0,0)_{0,0}+(1,0)_{0,0}+(0,1)_{0,0}+ \\ & & & [(\tfrac{1}{2},0)_{\pm1,0}+(0,\tfrac{1}{2})_{0,\pm1}] \\ II & (\mathbf{4},\mathbf{4}) & \downarrow & (\tfrac{1}{2},\tfrac{1}{2})_{0,0}+(0,0)_{\pm\frac{1}{2},\pm\frac{1}{2}}+(0,0)_{\pm\frac{1}{2},\mp\frac{1}{2}}+ \\ & & & [(\tfrac{1}{2},0)_{\pm\frac{1}{2},0}+(0,\tfrac{1}{2})_{0,\pm\frac{1}{2}}] \end{array} \qquad (9)$$

which precisely coincide with the component field content of the corresponding schizofield classes[‖].

Explicitly, consider $sl(2/1)\simeq osp(2/2)$ spanned by $\{K_{ab},\ Q_{a\pm},\ Y,\ a,b=1,2\}$:

$$\begin{array}{rcl} [K_{ab},K_{cd}]_- & = & \varepsilon_{cb}K_{ad}+\varepsilon_{ca}K_{bd}+\varepsilon_{da}K_{bc}+\varepsilon_{db}K_{ac} \\ [K_{ab},Q_{c\pm}]_- & = & \varepsilon_{cb}Q_{a\pm}+\varepsilon_{ca}Q_{b\pm} \\ [Y,Q_{a\pm}]_- & = & \pm\frac{1}{2}Q_{a\pm} \\ \{Q_{a\pm},Q_{b\pm}\}_+ & = & -K_{ab}+2\varepsilon_{ab}Y \end{array}$$

The super-Lorentz algebra $sl(2/1,\mathbb{C})_{\mathbb{R}}$ is spanned by combinations such as $\frac{1}{2}(K\pm iK')$ and so on, giving commuting left and right parts $J_{ab},\ S_{a\pm},\ Z^L;\ \bar{J}_{\dot{a}\dot{b}},\ \bar{S}_{\dot{a}},\ Z^R$ with the

[¶] As usual, $sl(2/1,\mathbb{C})\sim sl(2/1)^L+sl(2/1)^R$.

[‖] The type *III* schizofields are associated with an embedding of the Lorentz algebra in the orthosymplectic superalgebra $osp(3,1/4,\mathbb{R})$.

same algebra as above. Explicit matrix elements for representations of $sl(2/1)$ can be derived by acting the above generators on an abstract basis. In particular, the adjoint representation $\mathbf{8}$ follows from the structure constants. With basis $\{\Theta_{ab},\ \Theta,\ \vartheta_{a\pm}\}$ the action is:

$$
\begin{aligned}
K_{ab}\cdot\Theta_{cd} &= \varepsilon_{cb}\Theta_{ad}+\varepsilon_{ca}\Theta_{bd}+\varepsilon_{da}\Theta_{bc}+\varepsilon_{db}\Theta_{ac}\\
K_{ab}\cdot\vartheta_{c\pm} &= \varepsilon_{cb}\vartheta_{a\pm}+\varepsilon_{ca}\vartheta_{b\pm}\\
Y\cdot\vartheta_{a\pm} &= \pm\tfrac{1}{2}Q_{a\pm}\\
Q_{a\pm}\cdot\Theta_{bc} &= \varepsilon_{ba}\vartheta_{c\pm}+\varepsilon_{ca}\vartheta_{b\pm}\\
Q_{a\pm}\cdot\Theta &= \mp\tfrac{1}{2}\vartheta_{a\pm}\\
Q_{a\pm}\cdot\vartheta_{b\pm} &= 0\\
Q_{a\pm}\cdot\vartheta_{b\mp} &= -\Theta_{ab}+\varepsilon_{ab}\Theta.
\end{aligned}
\tag{10}
$$

In order to establish how the supersymmetry acts on type I and II schizofields, and thus explicitly establish Φ_I and Φ_{II} as $(\mathbf{8},0)+(0,\mathbf{8})$ and $(\mathbf{4},\mathbf{4})$ representations respectively, a concrete realisation of such modules must be given at the superfield level. The previously identified θ-polynomials (table 1) are equivalent to the abstract basis vectors, and appropriately defined operators $p(\theta)q(\partial)$ having the appropriate matrix elements establish the embedding of $sl(2/1)$ in $gl(8/8)$.

The procedure is illustrated for the $(\mathbf{8},0)+(0,\mathbf{8})$, with the $(\mathbf{4},\mathbf{4})$ left as a similar calculation. Firstly, the identification of the component field content of Φ_I (see (8)) must be completed by assigning two additional additive quantum numbers consistently with (9). However, from the right-hand column of table 1, it can be seen that the correct choice is to take these as magnetic quantum numbers from the *commuting* $sl(2)$ factors in each sector (^{L} or ^{R}, respectively), so that the additional abelian generators are

$$
\begin{aligned}
Z^L &= \bar{J}^-_{\dot{1}\dot{2}}\,\mathbb{P}^o,\\
Z^R &= J^-_{12}\,\mathbb{P}^o.
\end{aligned}
\tag{11}
$$

For a basis set corresponding to (10), define the following polynomials in θ^μ (see table 1 and associated text)

$$
\begin{aligned}
\Theta &= \tfrac{1}{2}(1-i\theta^4)\\
\Theta_{ab} &= \tfrac{1}{2}(\theta^{\mu\nu}+i\tilde{\theta}^{\mu\nu})(\sigma_{\mu\nu})_{ab}\\
\vartheta_{a+} &= \tfrac{1}{2}(\theta^\mu+i\tilde{\theta}^\mu)\sigma^\mu_{\dot{1}a}\\
\vartheta_{a-} &= \tfrac{1}{2}(\theta^\mu+i\tilde{\theta}^\mu)\sigma^\mu_{\dot{2}a}
\end{aligned}
\tag{12}
$$

together with the corresponding conjugates $\bar{\Theta}_{\dot{a}\dot{b}}, \bar{\Theta}, \bar{\vartheta}_{\dot{a}\alpha}$. Further introduce the corresponding polynomials in ∂^θ_μ, namely $\Delta_{ab}, \Delta, \varrho_{a\dot{\alpha}}$, say, and their conjugates:

$$
\begin{aligned}
\Delta &= \tfrac{1}{2}(1-i\partial^4)\\
\Delta_{ab} &= \tfrac{1}{2}(\partial^{\mu\nu}+i\tilde{\partial}^{\mu\nu})(\sigma_{\mu\nu})_{ab}\\
\varrho_{a+} &= \tfrac{1}{2}(\partial^\mu+i\tilde{\partial}^\mu)\sigma^\mu_{\dot{1}a}\\
\varrho_{a-} &= \tfrac{1}{2}(\partial^\mu+i\tilde{\partial}^\mu)\sigma^\mu_{\dot{2}a}
\end{aligned}
\tag{13}
$$

As has been pointed out by Eyal[7] (see also [8]), in the superfield context, a basis of $gl(8/8)$ corresponding to elementary matrices must be introduced with the use of appropriate zero projector $\mathbb{P}^{(0)}$ (see (6) above). For polynomials $q(\theta)$, $p(\theta)$, set

$$E_{pq} = p(\theta)\mathbb{P}^{(0)}q(\partial)$$

which maps between q and p (*with all other matrix elements zero*). Hence the schizosymmetric assignment of physical Lorentz generators on class I superfields is completed by the following generators of the super-Lorentz algebra establishing the embedding of the latter in $gl(8/8)$,

$$\begin{aligned}
Q_{a\pm} &= (-\Theta_{ab} + 2\varepsilon_{ab}\Theta)\mathbb{P}^{(0)}\varrho^{b\mp} + \vartheta^{b\pm}\mathbb{P}^{(0)}(2\varepsilon_{ab}\Delta - \Delta_{ab});\\
\overline{Q}_{\dot{a}\pm} &= (-\overline{\Theta}_{\dot{a}\dot{b}} + 2\varepsilon_{\dot{a}\dot{b}}\overline{\Theta})\mathbb{P}^{(0)}\overline{\varrho}^{\dot{b}\mp} + \overline{\vartheta}^{\dot{b}\pm}\mathbb{P}^{(0)}(2\varepsilon_{\dot{a}\dot{b}}\overline{\Delta} - \overline{\Delta}_{\dot{a}\dot{b}});\\
Z^L &= \bar{J}^-_{\dot{1}\dot{2}}\,\mathbb{P}^o;\\
Z^R &= J^-_{12}\,\mathbb{P}^o;\\
L^I_{\mu\nu} &= L^{diag}_{\mu\nu}\mathbb{P}^e + L^+_{\mu\nu}\mathbb{P}^o
\end{aligned}$$

4 Conclusions and outlook

In this paper the ideas of [1] on schizosymmetry in spacetime have been developed in detail, and the connection with $sl(2/1,\mathbb{C})_{\mathbb{R}}$ super-Lorentz symmetry at the component field level has been demonstrated for two of the three types of schizofield identified (for the third type, the supersymmetry is of orthosymplectic type $osp(3,1/4,\mathbb{R})$[9], but was not considered further).

The question of the physical status of such book-keeping supersymmetries remains to be established by further extending the formalism to a schizofield calculus. A major technical difficulty for the latter is that schizofield products are not closed as to type; a bilocal calculus

$$\Phi \star \Phi'(\theta) \sim \int_{\vartheta} \Phi(\theta - \vartheta)\Phi'(\vartheta)K(\theta,\vartheta)d\vartheta$$

may be required for covariance, and is under study**. From the super-Lorentz representations established in §3, namely $\Phi_I \simeq (\mathbf{8},0)+(0,\mathbf{8})$, and $\Phi_{II} \simeq (\mathbf{4},\mathbf{4})$, there is a natural identification of type II schizofields as 'gauge potential-like' (containing a vector potential A_μ as one component), and type I schizofields as 'field strength-like' (containing an antisymmetric tensor $F_{\mu\nu}$ as one component), respectively. A scenario for Lagrangian construction would then be to model a generalised gauge potential as a Grassmann-odd, type $\widetilde{II}$ schizofield $\widetilde{\Phi}_{II} \equiv \mathcal{A}$, say, and to introduce an 'exterior' operator of the form $\mathcal{D} = \Gamma^\mu\partial^x_\mu$ for suitable odd Γ^μ, for example $\Gamma^\mu = \theta^\mu$ or $\Gamma^\mu = \gamma^\mu$ (see (3)). Local gauge invariance would then be implemented through $\mathcal{F} = \mathcal{D}\mathcal{A} + \mathcal{A}\star\mathcal{A}$ (including possible nonabelian extensions)††. From this point of view, the present approach can be seen to address the question of supersymmetric generalisations of the Dirac operator in higher spin wave equations.

** The embedding of the super-Lorentz algebra in $gl(8/8)$ is not regular, leading to non-linear θ and ∂^θ terms in the generators, so that there is no Leibniz property for handling products of schizofields.

†† Note for example that $\theta^\mu\partial^x_\mu\theta^\nu A_\nu = \frac{1}{2}\theta^\mu\theta^\nu F_{\mu\nu}$

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